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Abstract

We construct a solution to a complex nonlinear heat equation which blows up in finite time T only at one blow-up point. We also give a sharp description of its blow-up profile. The proof relies on the reduction of the problem to a finite dimensional one and the use of index theory to conclude. We note that the real and imaginary parts of the constructed solution blow up simultaneously.

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1 Introduction

This paper is concerned with blow-up solutions of the complex heat equation

$$\partial_t u = \Delta u + u^2, \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to \mathbb{C}$, Δ denotes the Laplacian.
If we write $u(x,t) = v(x,t) + i\tilde{v}(x,t)$, where v and $\tilde{v} \in \mathbb{R}$, we will consider the following reaction-diffusion system.

$$\begin{array}{rcl} \partial_t v &=& \Delta v + v^2 - \tilde{v}^2, \\ \partial_t \tilde{v} &=& \Delta \tilde{v} + 2v\tilde{v}, \end{array} \tag{2}$$

where $(x,t) \in \mathbb{R}^N \times (0,T)$, $v(0,x) = v_0(x)$ and $\tilde{v}(0,x) = \tilde{v}_0(x)$.
The equation (1) has a strong relation with the viscous Constantin-Lax-Majda equation, which is a one dimensional model for the vorticity equation. For more details see Okamoto, Sakajo and Wunsch [OSW08], Sakajo [Sak03a] and [Sak03b] and Guo, Ninomiya, Shimojo and Yanagida in [GNSY13].

The Cauchy problem for system (2) can be solved in $(L^\infty(\mathbb{R}^N))^2$, locally in time. We say that $u(t) = v(t) + i\tilde{v}(t)$ blows up in finite time $T < \infty$, if $u(t)$ exists for all $t \in [0,T)$ and $\lim_{t\to T} \|v(t)\|_{L^\infty} + \|\tilde{v}(t)\|_{L^\infty} = +\infty$. In that case, T is called the blow-up time of the solution. A point $x_0 \in \mathbb{R}^N$ is said to be a blow-up point if there is a sequence $\{(x_j, t_j)\}$, such that $x_j \to x_0$, $t_j \to T$ and $|v(x_j,t_j)| + |\tilde{v}(x_j,t_j)| \to \infty$ as $j \to \infty$. The set of all blow-up points is called the blow-up set.

When u is real (i.e., $\tilde{v} \equiv 0$), then this system is reduced to the scalar equation

$$\partial_t u = \Delta u + u^p, \text{ where } p = 2. \tag{3}$$

The blow-up question for equation (3), with $p > 1$, has been studied intensively by many authors and no list can be exhaustive. Nevertheless, let us just mention the work of [Bal77], [GK85], [GK87], [GK89], [HV93], [HV94], [MM04], [MM09], [MZ98], [MZ00], [Miz07] and [QS07].
Note that there is another complex generalization of the real case given in (3). Indeed, Filippas and Merle consider in [FM95]the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u \text{ with } u \in \mathbb{C} \text{ and } p > 1, \tag{4}$$

and generalize to the complex case the results first proved in the real case by Giga and Kohn [GK85, GK87, GK89]. Our equation (1) appears as in the "twin" equation (4), however there is a fundamental difference between the two. Indeed, equation (4) has a variational structure, which allows to use various energy techniques, unlike equation (1), where such techniques certainly fail.
When u is not real, we have the following blow-up results from [GNSY13].

(A) A non-simultaneous blow-up criterion, see Theorem 1.5 in [GNSY13]:
Assume that

$$v_0,\ \tilde{v}_0 \in C^1(\mathbb{R}^m),\ 0 \le v_0 \le M,\ v_0 \not\equiv M,\ 0 < \tilde{v}_0 \le L, \tag{5}$$

$$\lim_{|x|\to\infty} v_0(x) = M, \quad \lim_{|x|\to\infty} \tilde{v}_0(x) = 0, \tag{6}$$

for some constants $L > 0$ and $M > 0$. Then, the solution of (2) blows up at time $t = T(M)$ with $\tilde{v} \neq 0$. Moreover, the component v blows up only at space infinity and $\tilde{v}$ remains bounded.

(B) A Fourier-based blow-up criterion, see Theorem 1.2 in [GNSY13]:
If the Fourier transform of initial data of (1) is real and positive, then the solution blows up.

(C) A simultaneous blow-up criterion, see Theorem 1.3 in [GNSY13]:
If $N = 1$, v_0 is even, $\tilde{v}_0$ is odd with $\tilde{v}_0(x) > 0$ for $x > 0$, then the fact that the blow-up set is compact implies that v and $\tilde{v}$ blow up simultaneously.

Unfortunately, in [GNSY13], the blow-up profile derivation remained open, apart of course from the trivial case where $\tilde{v} \equiv 0$ and where we know from Herrero and Velázquez [HV92] and [HV94] that generically, the blow-up set is reduced to a single point and

$$u(x,t) \sim (T-t)^{-1} f\left(\frac{x}{\sqrt{(T-t)|\log(T-t)|}}\right), \tag{7}$$

where

$$f(z) = \left(1 + \frac{1}{8}|z|^2\right)^{-1}, \text{ for } z \in \mathbb{R}^N. \quad (8)$$

Note that the proof of the genericity of (7) in higher dimensions has been announced by Herrero and Velázquez, however, they never publish it. Note also that the stability of such a profile with respect to initial data has been proved by Fermanian Kammerer, Merle and Zaag in [MZ97] and [FKMZ00].
In [EZ11], Ebde and Zaag show the persistence of this profile under perturbations of equation (1) in the real case by lower order terms involving u and ∇u.

In this paper, we go further towards the proof of a kind of structural stability result for the profile (8) and show the existence of a complex-valued solution to (1) obeying the behavior (8), and with non-zero $\text{Im } u \equiv \tilde{v}$. Let us note that the blow-up behavior we give here is not predicted by [GNSY13] (see details in the remarks following our result). More precisely, this is our result:

Theorem 1 ***(Existence of a blow-up solution for equation (1) with the description of its profile)****. There exists $T > 0$ such that equation (1) has a solution $u(x,t) = v(x,t) + i\tilde{v}(x,t)$ in $\mathbb{R}^N \times [0,T)$ such that:*
(i) the solution u blows up in finite time T only at the origin.
(ii) It holds that

$$\left\|(T-t)u(.,t) - f\left(\frac{.}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty} \leq \frac{C}{\sqrt{|\log(T-t)|}}, \quad (9)$$

where f is defined by (8).
(iii) For all $R > 0$

$$\sup_{|x| \leq R\sqrt{T-t}} \left|(T-t)\tilde{v}(x,t) - \frac{\sum_{i=1}^N C_i}{|\log(T-t)|^2}\left(\frac{x_i^2}{T-t} - 2\right)\right| \leq \frac{C}{|\log(T-t)|^\alpha}, \quad (10)$$

where $(C_1, C_2, .., C_N) \neq (0, 0, .., 0)$, for some small $\varepsilon > 0$.
(iv) For all $x \neq 0$, $u(x,t) \to u^*(x)$ uniformly on compacts sets of $\mathbb{R}^N \backslash \{0\}$, and

$$u^*(x) \sim \frac{16|\log|x||}{|x|^2} \text{ as } x \to 0. \quad (11)$$

Remarks:

1) Note that the real and imaginary parts of u blow up simultaneously at $x = 0$. However the real part dominates the imaginary part in the sense that

$$v(0,t) \sim \frac{1}{T-t} >> \frac{-2\sum_{i=1}^N C_i}{(T-t)|\log(T-t)|^2} \sim \tilde{v}(0,t) \text{ as } t \to \infty.$$

2) As announced right before the statement of our theorem, the solution we construct is new and doesn't obey the criteria given in [GNSY13] . Indeed, from (25) below, one can see that (5) and (6) are satisfied expect for the conditions on $\tilde{v}_0$. Indeed, $\tilde{v}_0$ changes sign and

can not be odd. The proof relies on the reduction of the problem to a $2(N+1)-$dimensional problem (a $4-$dimensional one if $N=1$; see subsection 3.4 below). In the real case treated by Merle and Zaag in [MZ97], our problem was of dimension $N+1$. Since that number is equal to the dimension of the blow-up parameters (1 for the blow-up time and N for the blow-up point), the authors of [MZ97] where able to show the stability of the behavior (9) with respect to initial data, of course in the real case. Here, in the complex case, since the dimension of our problem (2(N+1)) exceeds that of the blow-up parameters ($N+1$), we suspect our solution to be unstable with respect to perturbations in initial data.

Our proof uses some ideas developed by Bricmont and Kupiainen [BK94] and Merle and Zaag [MZ97] for the semilinear heat equation (3). In [MZ08], Masmoudi and Zaag adapted that method to the case of the following complex Ginzburg-Landau equation, where no gradient structure exists:

$$\partial_t u = (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u, \text{ where } \beta \text{ and } \delta \text{ are reals,}$$

(note that the case $\beta = 0$ and δ small was studied by Zaag in [Zaa98]).
More precisely, the proof relies on the understanding of the dynamics of the selfsimilar version of (2) (see system (14) below) around the profile (8). Moreover, we proceed in two steps:

- In Step 1, we reduce the question to a finite-dimensional problem: we show that it is enough to control a $(N+1)$-dimensional variable in order to control the solution (which is infinite dimensional) near the profile.
- In Step 2, we proceed by contradiction to solve the finite-dimensional problem and conclude using index theory.

Surprisingly enough, we would like to mention that this kind of methods has proved to be successful in various situations including hyperbolic and parabolic PDE, in particular with energy critical exponents. This was the case for the construction of multi-solitons for the semilinear wave equation in one space dimension by Côte and Zaag [CZ13], the wave maps by Raphaël and Rodnianski [RR12], the Schrödinger maps by Merle, Raphaël and Rodnianski [MRR11], the critical harmonic heat flow by Schweyer [Sch12] and the two-dimensional Keller-Segel equation by Raphaël and Schweyer [RS13].

We proceed in 4 sections to prove Theorem 1. We first give in Section 2 an equivalent formulation of the problem in the scale of the well-known similarity variables. Section 3 is devoted to the proof of the similary variables formulation (this is a central part in our argument). Finally, we conclude the proof of Theorem 1 in Section 4.

2 Formulation of the problem

For simplicity, we give the proof in one dimension. The adaptation to higher dimensions is straightforward. We would like to find initial data $u_0 = v_0 + i\tilde{v}_0$ such that the solution $u = v + i\tilde{v}$ of equation (2) blows up in time T and

$$\lim_{t\to T} \left\| (T-t)u(x,t) - f\left(\frac{x}{\sqrt{(T-t)|\log(T-t)|}}\right) \right\|_{L^\infty} = 0. \tag{12}$$

This is the main estimate and the other results of Theorem 1 will appear as by products of the proof.
Given an arbitrary $T>0$, we introduce the following self-similar transformation of problem (2)

$$\begin{array}{c} w(y,s)=(T-t)v(x,t),\ \tilde w(y,s)=(T-t)\tilde v(x,t), \\ y=\frac{x-a}{\sqrt{T-t}},\ s=-\log(T-t). \end{array} \tag{13}$$

If $(v,\tilde v)$ is a solution of (2), then the function $(w=w_a,\tilde w=\tilde w_a)$ satisfies for all $s\geq -\log T$ and $y\in\mathbb{R}$:

$$\begin{array}{rcl} \partial_s w &=& \partial_y^2 w-\frac{1}{2}y\cdot\partial_y w-w+w^2-\tilde w^2, \\ \partial_s\tilde w &=& \partial_y^2\tilde w-\frac{1}{2}y\cdot\partial_y\tilde w-\tilde w+2w\tilde w. \end{array} \tag{14}$$

Using the selfsimilar variables, (12) is equivalent to finding $s_0>0$ and initial data at s_0, $W_0(y,s_0)=w_0(y,s_0)+i\tilde w_0(y,s_0)$, such that the solution of (14) $W(y,s)=w(y,s)+i\tilde w(y,s)$ satisfies

$$\lim_{s\to\infty}\left\|W(y,s)-f\left(\frac{y}{\sqrt{s}}\right)\right\|_{L^\infty}=0. \tag{15}$$

Introducing

$$w=\varphi+q \text{ and } \tilde w=\tilde q \text{ where } \varphi(y,s)=f\left(\frac{y}{\sqrt{s}}\right)+\frac{1}{4s}, \tag{16}$$

the problem is then reduced to constructing a function $Q=q+i\tilde q$ such that

$$\lim_{s\to\infty}\|Q(y,s)\|_{L^\infty}=0,$$

and $(q,\tilde q)$ is a solution of the following equation for all $(y,s)\in\mathbb{R}\times[s_0,\infty)$,

$$\begin{array}{rcl} \partial_s q &=& (\mathcal{L}+V)q+B(y,s)-N(y,s)+R(y,s), \\ \partial_s\tilde q &=& (\mathcal{L}+V)\tilde q+\tilde B(y,s), \end{array} \tag{17}$$

where

$$\mathcal{L}=\partial_y^2-\frac{1}{2}y\cdot\partial_y+1,\ V(y,s)=2\left(\varphi(y,s)-1\right), \tag{18}$$

$$B(y,s)=q^2,\ N(y,s)=\tilde q^2,\ \tilde B(y,s)=2q\tilde q, \tag{19}$$

and

$$R(y,s)=\partial_y^2\varphi-\frac{1}{2}y\cdot\partial_y\varphi-\varphi+\varphi^2-\partial_s\varphi. \tag{20}$$

We introduce also the Hilbert space

$$L^2_\rho=\{g\in L^2_{loc}(\mathbb{R},\mathbb{C}),\ \|g\|^2_{L^2_\rho}\equiv\int_{\mathbb{R}}|g|^2e^{-\frac{|y|^2}{4}}dy<+\infty\} \text{ where } \rho(y)=\frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{1/2}}.$$

The operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R})$. The spectrum of $\mathcal{L}$ is explicitly given by

$$spec(\mathcal{L})=\{1-\frac{m}{2},\ m\in\mathbb{N}\}.$$

All the eigenvalues are simple and the eigenfunctions are dilations of Hermite's polynomial and given by

$$h_m(y)=\sum_{n=0}^{[\frac{m}{2}]}\frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{21}$$

We also introduce k_m,

$$k_m(y) = \frac{h_m(y)}{\|h_m(y)\|^2_{L^2_\rho}}. \tag{22}$$

Note that $\mathcal{L}$ has two positive (or expanding) directions ($\lambda = 1$ and $\lambda = \frac{1}{2}$), and a zero direction ($\lambda = 0$). Considering the fact that the aimed behavior in (15) shows a free boundary moving like $\sqrt{s}$, we decompose q and $\tilde{q}$ as follows:
Let us consider a non-increasing cut-off function $\chi_0 \in C_0^\infty(\mathbb{R}^+, [0,1])$ such that $supp(\chi_0) \subset [0,2]$, $\chi_0(\xi) = 1$ for $\xi < 1$ and $\chi_0(\xi) = 0$ for $\xi > 2$ and introduce

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right),$$

where $K_0 \geq 1$ will be chosen large enough so that various technical estimates hold.
We write $q = q_b + q_e$ and $\tilde{q} = \tilde{q}_b + \tilde{q}_e$, where the inner parts and the outer parts are given by

$$q_b = q\chi,\ \tilde{q}_b = \tilde{q}\chi,\ q_e = q(1-\chi) \text{ and } \tilde{q}_e = \tilde{q}(1-\chi).$$

Let us remark that

$$supp(q_b(s)) \subset B(0, 2K_0\sqrt{s}) \text{ and } supp(q_e(s)) \subset \mathbb{R} \setminus B(0, K_0\sqrt{s}).$$

Then, we study q_b and $\tilde{q}_b$ using the structure of $\mathcal{L}$, isolating the nonnegative directions. More precisely we decompose q_b and $\tilde{q}_b$ as follows

$$\begin{array}{rcl} q_b(y,s) &=& \sum_0^2 q_m(s)h_m(y) + q_-(y,s), \\ \tilde{q}_b(y,s) &=& \sum_0^2 \tilde{q}_m(s)h_m(y) + \tilde{q}_-(y,s), \end{array} \tag{23}$$

where q_m (respectively $\tilde{q}_m$) is the projection of q_b (respectively $\tilde{q}_b$) on h_m and $q_-(y,s) = P_-(q_b)$ (respectively $\tilde{q}_-$) and P_- is the projection in the negative subspace of the $\mathcal{L}$. Thus, we can decompose q (respectively $\tilde{q}$) in 5 components as follows:

$$\begin{array}{rcl} q(y,s) &=& \sum_{m=0}^2 q_m(s)h_m(y) + q_-(y,s) + q_e(y,s), \\ \tilde{q}(y,s) &=& \sum_{m=0}^2 \tilde{q}_m(s)h_m(y) + \tilde{q}_-(y,s) + \tilde{q}_e(y,s). \end{array} \tag{24}$$

Here and throughout the paper, we call $q_-(y,s)$ (respectively $\tilde{q}_-$) the negative part of q (respectively $\tilde{q}$), q_0 (respectively $\tilde{q}_0$), the null mode of q (respectively $\tilde{q}$), and the subspace spanned by $\{h_m, m \geq 3\}$ will be referred to as the negative subspace.

3 The proof in selfsimilar variables

This section is devoted to the proof of the existence of a solution $(q, \tilde{q})$ of system (17) satisfying $\|q(s)\|_{L^\infty} + \|\tilde{q}(s)\|_{L^\infty} \to 0$. This is a central argument in our proof. In Section 4, we use this solution and give the proof of Theorem 1. We proceed in 5 steps, each of them making a separate subsection. Note that our argument is derived from the work of Merle and Zaag in [MZ97]. For that reason, we will stress only the main parts of the proof and put forward the novelties of our argument. In particular, we will avoid purely technical details and refer the interested reader to specific statements in [MZ97].

In the first subsection, we define shrinking sets $V_A(s)$ and $\tilde{V}_{\tilde{A}}(s)$ and translate our goal of making $(q(s), \tilde{q}(s))$ go to $(0,0)$ in $L^\infty(\mathbb{R})$ in terms of belonging to $V_A(s) \times \tilde{V}_A(s)$. We state this goal in Proposition 3.3 below, the following parts of this section are devoted to the proof of that proposition.
In the second subsection, we solve the local in time Cauchy problem.
In the third subsection, we reduce our goal from the control of $(q(s), \tilde{q}(s))$ in $V_A(s) \times \tilde{V}_{\tilde{A}}(s)$ to the control of $(q_0, q_1, \tilde{q}_0, \tilde{q}_1)$ in $[-\frac{A}{s^2}, \frac{A}{s^2}]^2 \times [-\frac{\tilde{A}}{s^\alpha}, \frac{\tilde{A}}{s^\alpha}]^2$.
In the forth subsection, we solve the finite dimensional problem using the index theory and conclude the proof of Proposition 3.3.
In the last subsection, we give some links and details for the reduction to a finite-dimensional problem.

3.1 Definition of a shrinking set $V_A(s)$, $\tilde{V}_{\tilde{A}}(s)$ and preparation of initial data

Let us first introduce the following proposition:

Proposition 3.1 *(A set shrinking to zero) For all $A \geq 1$, $\tilde{A} \geq 1$ and $s \geq e$, we define $V_A(s)$ (respectively $\tilde{V}_{\tilde{A}}(s)$) as the set of all function r (respectively $\tilde{r}$) in L^∞ such that:*

$$|r_m(s)| \leq As^{-2} \; m = 0, 1, \;\; |r_2(s)| \leq A^2 (\log s) s^{-2},$$

$$\forall y \in \mathbb{R}, \; |r_-(y, s)| \leq A(1 + |y|^3) s^{-2}, \; \|r_e(s)\|_{L^\infty} \leq A^2 s^{-\frac{1}{2}},$$

and

$$|\tilde{r}_m(s)| \leq \tilde{A} s^{-\alpha} \; m = 0, 1, \;\; |\tilde{r}_2(s)| \leq \tilde{A}^2 s^{-2+\varepsilon},$$

$$\forall y \in \mathbb{R}, \; |\tilde{r}_-(y, s)| \leq \tilde{A}(1 + |y|^3) s^{-\alpha}, \; \|\tilde{r}_e(s)\|_{L^\infty} \leq \tilde{A}^2 s^{-\alpha+3/2},$$

where r_-, r_e and r_m are defined in (24) and $2 < \alpha \leq 2 + \varepsilon$. Then, for all $s \geq e$, $r \in V_A(s)$ and $\tilde{r} \in \tilde{V}_{\tilde{A}}(s)$, we have

$$(i) \; for \; all \; y \in \mathbb{R}, \;\; |r(y, s)| \leq CA^2 \frac{\log s}{s^2} (1 + |y|^3), \qquad (ii) \|r(s)\|_{L^\infty} \leq C \frac{A^2}{\sqrt{s}},$$

$$(iii) \; for \; all \; y \in \mathbb{R}, \;\; |\tilde{r}(y, s)| \leq C\tilde{A}^2 \frac{1}{s^{2-\varepsilon}} (1 + |y|^3), \;\; |\tilde{r}_b(y, s)| \leq \frac{C\tilde{A}}{s^{\alpha-3/2}}, \quad (iv) \|\tilde{r}(s)\|_{L^\infty} \leq C \frac{\tilde{A}^2}{s^{\alpha-3/2}}.$$

Proof: The proof is omitted since it is the same as the corresponding part in [MZ97]. See Proposition 3.7 page 157 in [MZ97] for details. ■

Initial data (at time $s = s_0 = -\log T$) for the equation (17) will depend on five real parameters d_0, d_1, $\tilde{d}_0$, $\tilde{d}_1$ and $\tilde{d}_2$ as given in the following proposition.

Proposition 3.2 *(Decomposition of initial data in differents components.) For each $|\tilde{d}_2| \leq 1$, $A \geq 1$ and $\tilde{A} \geq 1$ there exists $T_1(A, \tilde{A}, \tilde{d}_2) \in (0, 1/e)$ such that for all $T \leq T_1$: If we consider as initial data for the equation (17) the following functions:*

$$\begin{aligned} q_{d_0,d_1}(y, s_0) &= \frac{A}{s_0^2}(d_0 + d_1 y)\chi(2y, s_0), \\ \tilde{q}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(y, s_0) &= \left[\frac{\tilde{A}}{s_0^\alpha}(\tilde{d}_0 + \tilde{d}_1 y) + \frac{\tilde{d}_2}{s_0^2} h_2(y)\right] \chi(2y, s_0), \end{aligned} \tag{25}$$

where $s_0 = -\log T$, then,
(i) there exists a cuboid

$$D_T \subset [-2,2]^4, \tag{26}$$

such that the mapping $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \to (q_0(s_0), q_1(s_0), \tilde{q}_0(s_0), \tilde{q}_1(s_0))$ is linear and one to one from D_T onto $[-\frac{A}{s_0^{2-\varepsilon}}, \frac{A}{s_0^{2-\varepsilon}}]^2 \times [-\frac{\tilde{A}}{s_0^{\alpha}}, \frac{\tilde{A}}{s_0^{\alpha}}]^2$ and maps ∂D_T into $\partial \left([-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2 \times [-\frac{\tilde{A}}{s_0^{\alpha}}, \frac{\tilde{A}}{s_0^{\alpha}}]^2\right)$. Moreover, it is of degree one on the boundary.
(ii) For all $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$, we have

$$\begin{aligned} &|q_2(s_0)| \leq CAe^{-\gamma s_0}, \text{ for some } \gamma > 0, \ \ |q_-(y,s_0)| \leq \tfrac{c}{s_0^2}(1+|y|^3) \text{ and } q_e(y,s_0) = 0, \\ &|d_0| + |d_1| \leq 1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &|\tilde{q}_2(s_0) - \tfrac{\tilde{d}_2}{s_0^2}| \leq C\tilde{A}e^{-\gamma s_0}, \text{ for some } \gamma > 0, \ \ |\tilde{q}_-(y,s_0)| \leq \tfrac{c}{s_0^{\alpha}}(1+|y|^3) \text{ and } \tilde{q}_e(y,s_0) = 0, \\ &|\tilde{d}_0| + |\tilde{d}_1| \leq 1. \end{aligned} \tag{28}$$

(iii) For all $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$, $q(s_0) = q_{d_0,d_1}(s_0) \in V_A(s_0)$, $\tilde{q}(s_0) = \tilde{q}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(s_0) \in \tilde{V}_{\tilde{A}}(s_0)$, with strict inequalities except for $(q_0(s_0), q_1(s_0), \tilde{q}_0(s_0), \tilde{q}_1(s_0))$, in the sense that

$$|q_m(s_0)| \leq As_0^{-2} \ m = 0,1, \ \ |q_2(s_0)| < A^2(\log s_0)s_0^{-2},$$

$$\forall y \in \mathbb{R}, \ |q_-(y,s_0)| < A(1+|y|^3)s_0^{-2}, \ \|q_e(s_0)\|_{L^\infty} < A^2 s_0^{-\frac{1}{2}},$$

and

$$|\tilde{q}_m(s_0)| \leq \tilde{A}s_0^{-\alpha} \ m = 0,1, \ \ |\tilde{q}_2(s_0)| < \tilde{A}^2 s_0^{-2+\varepsilon},$$

$$\forall y \in \mathbb{R}, \ |\tilde{q}_-(y,s_0)| < \tilde{A}(1+|y|^3)s_0^{-\alpha}, \ \|\tilde{q}_e(s_0)\|_{L^\infty} < \tilde{A}^2 s_0^{-\alpha+\frac{3}{2}}.$$

Proof: Since we have almost the same definition of the set V_A, and almost the same expression of initial data $q(d_0)$ as in [MZ97], we refer the reader to Lemma 3.5 and Lemma 3.9 from [MZ97]. ■

In this section, we will prove the following proposition, which directly implies Proposition 3.2 thanks to Proposition 3.1:

Proposition 3.3 *There exists A_0 such that for all $A \geq A_0$ and $\tilde{A} \geq A_0$, there exists $T_0(A, \tilde{A})$ such that for all $T \leq T_0$ and $|\tilde{d}_2| \leq 1$, there exists $(d_0, d_1, \tilde{d}_0, \tilde{d}_1)$, such that, if $(q, \tilde{q})$ is a solution of (17) with initial data at $s_0 = -\log T$ given by (25), then*

$$\forall s \geq -\log T, \ \ q(s) \in V_A(s) \text{ and } \tilde{q}(s) \in \tilde{V}_{\tilde{A}}(s).$$

The remaining part of the section is devoted to the proof of this proposition.

3.2 Local in time solution for equation (17)

In the following, we find a local in time solution for equation (17).

Proposition 3.4 *(Local in time solution for equation (17) with initial data (25)) For all $A \geq 1$ and $\tilde{A} \geq 1$, there exists $T_2(A, \tilde{A}) \leq T_1(A, \tilde{A})$, such that for all $T \leq T_2$, the following holds:*
For all $|\tilde{d}_2| \leq 1$ and $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$ with initial data at $s = s_0$, $(q_{d_0,d_1}(s_0), \tilde{q}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(s_0))$ defined in (25), there exists a $s_{max} > s_0$ such that equation (17) has a unique solution satisfying $(q(s), \tilde{q}(s)) \in V_{A+1}(s) \times \tilde{V}_{\tilde{A}+1}(s)$ for all $s \in [s_0, s_{max})$.

Proof: From the definition of q in (16) we can see that the Cauchy problem of (17) is equivalent to that of equation (14) which is equivalent to the Cauchy problem of equation (2). Moreover, the initial data $((q_{d_0,d_1}(s_0), \tilde{q}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(s_0)))$ defined in (25) gives the following initial data for (2)

$$\begin{aligned} v_{d_0,d_1}(x) &= T^{-1}\left\{\frac{A}{(\log T)^2}(d_0 + d_1 y)\chi\left(\frac{2x}{\sqrt{T}}, -\log T\right) + \varphi(z, -\log T)\right\}, \\ \tilde{v}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(x) &= T^{-1}\left\{\frac{A}{|\log T|^\alpha}(\tilde{d}_0 + \tilde{d}_1 y) + \frac{\tilde{d}_2}{(\log T)^2} h_2\left(\frac{x}{\sqrt{T}}\right)\right\}\chi\left(\frac{2x}{\sqrt{T}}, -\log T\right), \end{aligned} \tag{29}$$

where $z = x(|\log T|T)^{-1/2}$.These initial data belong to $(L^\infty(\mathbb{R}))^2$ which insures the local existence (see the introduction) of $(v, \tilde{v})$ in $(L^\infty(\mathbb{R}))^2$. Now, since we have from *(iii)* of Proposition 3.2, $(q_{d_0,d_1}(s_0), \tilde{q}_{\tilde{d}_0,\tilde{d}_1,\tilde{d}_2}(s_0)) \in V_A(s_0) \times \tilde{V}_{\tilde{A}}(s_0) \subseteq V_{A+1}(s_0) \times \tilde{V}_{\tilde{A}+1}(s_0)$, there exists s_3 such that for all $s \in [s_0, s_3)$, $(q(s), \tilde{q}(s)) \in V_{A+1}(s) \times \tilde{V}_{\tilde{A}+1}(s)$. This concludes the proof of the proposition. ■

3.3 Reduction to a finite-dimensional problem

This step is crucial in the proof of Proposition 3.3. In this step, we will prove through a priori estimates that for each $s \geq s_0$, the control of $(q(s), \tilde{q}(s)) \in V_A(s) \times \tilde{V}_{\tilde{A}}(s)$ is reduced to the control of $(q_0(s), q_1(s), \tilde{q}_0(s), \tilde{q}_1(s)) \in \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^2 \times \left[-\frac{\tilde{A}}{s^\alpha}, \frac{\tilde{A}}{s^\alpha}\right]^2$. In fact, this result implies that if for some $s_1 \geq s_0$, $(q(s_1), \tilde{q}(s_1)) \in \partial\left(V_A(s_1) \times \tilde{V}_{\tilde{A}}(s_1)\right)$, then $(q_0(s_1), q_1(s_1), \tilde{q}_0(s_1), \tilde{q}_1(s_1)) \in \partial\left(\left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^2 \times \left[-\frac{\tilde{A}}{s^\alpha}, \frac{\tilde{A}}{s^\alpha}\right]^2\right)$.

Proposition 3.5 *(Control of $(q(s), \tilde{q}(s))$ by $(q_0(s), q_1(s), \tilde{q}_0(s), \tilde{q}_1(s))$ in $V_A(s) \times \tilde{V}_{\tilde{A}}(s)$.) There exists $A_3 > 0$ such that for each $A \geq A_3$ and $\tilde{A} \geq A_3$ there exists $T_3(A, \tilde{A}) \leq T_2(A, \tilde{A})$ such that for all $T \leq T_3$, the following holds:*
If $(q, \tilde{q})$ is a solution of (17) with initial data at $s = s_0 = -\log T$ given by (25) with $|\tilde{d}_2| \leq 1$, $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$, and $(q(s), \tilde{q}(s)) \in V_A(s) \times \tilde{V}_{\tilde{A}}(s)$ for all $s \in [s_0, s_1]$, with $(q(s_1), \tilde{q}(s_1)) \in \partial\left(V_A(s_1) \times \tilde{V}_{\tilde{A}}(s_1)\right)$ for some $s_1 \geq s_0$, then:
(i) (Reduction to a finite dimensional problem)
$(q_0(s_1), q_1(s_1), \tilde{q}_0(s_1), \tilde{q}_1(s_1)) \in \partial\left(\left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^2 \times \left[-\frac{\tilde{A}}{s^\alpha}, \frac{\tilde{A}}{s^\alpha}\right]^2\right)$.
(ii) (Transverse crossing) There exists m, $\tilde{m} \in \{0, 1\}$ and ω, $\tilde{\omega} \in \{-1, 1\}$ such that

$$\omega q_m(s_1) = \frac{A}{s_1^2} \text{ and } \omega \frac{dq}{ds}(s_1) > 0,$$

$$\tilde{\omega}\tilde{q}_{\tilde{m}}(s_1) = \frac{\tilde{A}}{s_1^{\alpha}} \text{ and } \tilde{\omega}\frac{dq\tilde{m}}{ds}(s_1) > 0,$$

Proof: Let us consider $A \geq 1$ and $T \leq T_2(A, \tilde{A})$. We then consider $(q, \tilde{q})$ a solution of (17) with initial data at $s = s_0 = -\log T$ given by (25) with $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$, and $(q(s), \tilde{q}(s)) \in V_A(s_1) \times \tilde{V}_{\tilde{A}}(s_1)$ for all $s \in [s_0, s_1]$, with $(q(s_1), \tilde{q}(s_1)) \in \partial\left(V_A(s_1) \times \tilde{V}_{\tilde{A}}(s_1)\right)$ for some $s_1 \geq s_0$. We now claim the following:

Proposition 3.6 *There exists $A_4 \geq 1$ such that for all $A \geq A_4$, $\tilde{A} \geq A_4$ and $\eta \geq 0$, there exists $T_4(A, \tilde{A}, \eta)$ such that the following holds for all $T \geq T_4(A, \tilde{A}, \eta)$:*
Assume that for some $\tau \geq s_0 = -\log T$ and for all $s \in [\tau, \tau + \eta]$,

$$(q(s), \tilde{q}(s)) \in V_A(s) \times \tilde{V}_A(s).$$

Then, the following holds for all $s \in [\tau, \tau + \eta]$:
(i)(Differential inequalities satisfied by the expanding and null modes) For $m = 0$ and 1, we have

$$\left|q_m'(s) - (1 - \frac{m}{2})q_m(s)\right| \leq \frac{C}{s^2},$$

$$\left|\tilde{q}_m'(s) - (1 - \frac{m}{2})\tilde{q}_m(s)\right| \leq \frac{C\tilde{A}^2}{s^{3-\varepsilon}},$$

$$\left|\tilde{q}_2'(s) + \frac{2}{s}\tilde{q}_2(s)\right| \leq \frac{C\tilde{A}}{s^{\alpha+1}}.$$

(ii)(Control of the null and negative modes) Moreover, we have

$$|q_2(s)| \leq \frac{\tau^2}{s^2}|q_2(\tau)| + \frac{CA(s-\tau)}{s^3},$$

$$|\tilde{q}_2(s)| \leq \frac{\tau^2}{s^2}|\tilde{q}_2(\tau)| + \frac{C\tilde{A}(s-\tau)}{s^{\alpha+1}},$$

$$\left\|\frac{q_-(s)}{1+|y|^3}\right\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{2}}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|q_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C(1+s-\tau)}{s^2},$$

$$\left\|\frac{\tilde{q}_-(s)}{1+|y|^3}\right\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{2}}\left\|\frac{\tilde{q}_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|\tilde{q}_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C(1+s-\tau)}{s^{\alpha}},$$

$$\|q_e(s)\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{2}}\|q_e(\tau)\|_{L^\infty} + C\frac{e^{s-\tau}}{s^{3/2}}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C(1+s-\tau)}{s^{1/2}},$$

$$\|\tilde{q}_e(s)\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{2}}\|\tilde{q}_e(\tau)\|_{L^\infty} + C\frac{e^{s-\tau}}{s^{3/2}}\left\|\frac{\tilde{q}_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C(1+s-\tau)}{s^{\alpha-3/2}}.$$

Proof: The proof is technical and long. For that reason, we leave it to Section 3.5 and proceed with the proof of Proposition of Proposition 3.5. ■

Now, we return to the proof of Proposition 3.5. Using Proposition 3.6, one can see that Proposition 3.5 follows exactly as in the case of semilinear heat equation treated in [MZ97]. The proof is easy, however a bit technical. That is the reason why it is omitted. The interested reader can find details in pages 160-164 of [MZ97] for (i), and in page 158 of [MZ97] for (ii). This concludes the proof of Proposition 3.5. ■

3.4 Proof of the finite dimensional problem

In this section, we give the proof of Proposition 3.3 (assuming that Proposition 3.6 holds, see section 3.5 for its proof). Although the derivation of Proposition 3.3 from Proposition 3.5 is the same as in [MZ97], we would like to give details for the reader's convenience, given that this is the heart of the proof and that it explains the two-point strategy: reduction to a finite dimensional problem and the proof of this problem using index theory.

Proof of Proposition 3.5:
Let us take $A = \tilde{A} \geq A_3$ and $T \leq T_3(A_3, A_3)$ given in Proposition 3.5. Consider $|\tilde{d}_2| \leq 1$. We proceed by contradiction and assume from (iii) of Proposition 3.2, that for all $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$, there exists $s_*(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \geq -\log T$, such that for all $s \in [-\log T, s_*]$,

$$(q, \tilde{q})_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(s) \in V_A \times \tilde{V}_{\tilde{A}}(s)$$

and

$$(q, \tilde{q})_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(s_*) \in \partial\left(V_A \times \tilde{V}_{\tilde{A}}(s_*)\right).$$

From (i) of Proposition 3.5, we see that

$$(q_0, q_1, \tilde{q}_0, \tilde{q}_1)_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(s_*) \in \partial\left(\left[-\frac{A}{s_*^2}, \frac{A}{s_*^2}\right]^2 \times \left[-\frac{\tilde{A}}{s_*^\alpha}, \frac{\tilde{A}}{s_*^\alpha}\right]^2\right)$$

and the following function is well defined:

$$\begin{aligned} \phi(y,s) \quad &: D_T \to \partial([-1,1]^4) \\ &(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \to s_*^2\left(\frac{q_0}{A}, \frac{q_1}{A}, \frac{\tilde{q}_0}{\tilde{A}}, \frac{\tilde{q}_1}{\tilde{A}}\right)_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(s_*). \end{aligned} \tag{30}$$

From the transverse crossing stated in (ii) of Proposition 3.5, ϕ is continuous. If we manage to prove that ϕ is of degree one on the boundary, then we have a contradiction from the degree theory. Let us prove that.
Using (i) and (iii) of Proposition 3.2 and the fact that

$$(q, \tilde{q})_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(-\log T) = (q_{d_0,d_1}, \tilde{q}_{\tilde{d}_0,\tilde{d}_1})(-\log T),$$

we see that if $(d_0, d_1, \tilde{d}_0, \tilde{d}_1)$ is in the boundary of the cuboid D_T, then

$$(q_0, q_1, \tilde{q}_0, \tilde{q}_1)_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(-\log T) \in \partial\left(\left[-\frac{A}{s_*^2}, \frac{A}{s_*^2}\right]^2 \times \left[-\frac{\tilde{A}}{s_*^\alpha}, \frac{\tilde{A}}{s_*^\alpha}\right]^2\right)$$

and $(q, \tilde{q})_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(-\log T) \in V_A \times \tilde{V}_{\tilde{A}}(-\log T)$, with strict inequalities for the other components. Applying the transverse crossing property of (ii) in Proposition 3.5, we see that $(q, \tilde{q})_{d_0,d_1,\tilde{d}_0,\tilde{d}_1}(s)$ leaves $V_A \times \tilde{V}_{\tilde{A}}(s)$ at $s = -\log T$, hence $s_* = -\log T$.
Using (i) of Proposition 3.2, we see that the restriction of ϕ to the boundary is of degree one. Since we know that ϕ is a continuous mapping from D_T to the boundary of $[-1,1]^4$, a contradiction then follows. Thus, there exists a value $(d_0, d_1, \tilde{d}_0, \tilde{d}_1) \in D_T$ (which depends on T and $\tilde{d}_2$) such that for all $s \geq -\log T$, $(q, \tilde{q})(s)_{d_0,d_1,\tilde{d}_0,\tilde{d}_1} \in V_A \times \tilde{V}_{\tilde{A}}(s)$. This concludes the proof of Proposition 3.3 assuming that Proposition 3.6 holds. ■

3.5 Proof of Proposition 3.6

We give the proof of Proposition 3.6 here. The proof consists in the projection of the two equations of system (17) on the different components of q and $\tilde{q}$ defined in (24).
When $\tilde{q} \equiv 0$, the proof is already available from Lemma 3.13 pages 167 from [MZ97].
When $\tilde{q} \not\equiv 0$, since the equation satisfied by $\tilde{q}$ in (17) shares the same linear part as the equation in q, the proof is similar to the argument in [MZ97]. For that reason, we only give the ideas here, and kindly ask the interested reader to look at Lemma 3.13 page 167 in [MZ97] for the technical details.

(i) Multiplying the equation in (17) by $\chi(y,s)k_m(y)\rho(y)$, for $m = 0, 1, 2$ and integrating in $y \in \mathbb{R}$, we proceed as in pages 158-159 from [MZ97] and we get the differential inequalities given in (i) with no difficulties.

(ii) We will find the main contribution in the projection given in the decomposition (24) of terms appearing in the right-hand side of equation (17). Let us first recall equations of $(q,\tilde{q})$ in their Duhamel formulation,

$$\begin{array}{rcl} q(s) & = & K(s,\tau)q(\tau) + \int_\tau^s d\sigma K(s,\sigma)B(q(\sigma)) + \int_\tau^s d\sigma K(s,\sigma)R(\sigma) - \int_\tau^s d\sigma K(s,\sigma)N(\sigma), \\ \tilde{q}(s) & = & K(s,\tau)\tilde{q}(\tau) + \int_\tau^s d\sigma K(s,\sigma)\tilde{B}(\sigma), \end{array} \tag{31}$$

where K is the fundamental solution of the operator $\mathcal{L} + V$. We write $q = \alpha + \beta + \gamma + \delta$ and $\tilde{q} = \tilde{\alpha} + \tilde{\beta}$, where

$$\begin{array}{l} \alpha(s) = K(s,\tau)q(\tau), \ \ \beta(s) = \int_\tau^s d\sigma K(s,\sigma)B(q(\sigma)), \\ \gamma(s) = \int_\tau^s d\sigma K(s,\sigma)R(\sigma), \ \ \delta(s) = -\int_\tau^s d\sigma K(s,\sigma)N(\sigma). \end{array} \tag{32}$$

$$\tilde{\alpha}(s) = K(s,\tau)\tilde{q}(\tau), \ \ \tilde{\beta}(s) = \int_\tau^s d\sigma K(s,\sigma)\tilde{B}(\sigma). \tag{33}$$

We assume that $(q(s), \tilde{q}(s)) \in V_A(s) \times \tilde{V}_A(s)$ for each $s \in [\tau, \tau + \eta]$. Clearly (ii) of Proposition 3.6 follows from the following:

Lemma 3.7 *There exists $A_5 \geq 1$ such that for all $A \geq A_5$, $\tilde{A} \geq A_5$, and $\eta > 0$ there exists $T_5(A, \tilde{A}, \eta) \leq T_2(A)$, such that for all $T \leq T_5(A, \tilde{A}, \eta)$, if we assume that for all $s \in [\tau, \tau + \eta]$, $q(s) \in V_A(s)$ and $\tilde{q}(s) \in \tilde{V}_{\tilde{A}}(s)$, then*
(i) (Linear terms)

$$\begin{array}{rcl} |\alpha_2(s)| & \leq & \frac{\tau^2}{s^2}|q_2(\tau)| + \frac{CA(s-\tau)}{s^3}, \\ \left\| \frac{\alpha_-(s)}{1+|y|^3} \right\|_{L^\infty} & \leq & Ce^{-\frac{(s-\tau)}{2}} \left\| \frac{q_-(\tau)}{1+|y|^3} \right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|q_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C}{s^2}, \\ \|\alpha_e(s)\|_{L^\infty} & \leq & Ce^{-\frac{(s-\tau)}{2}} \|q_e(\tau)\|_{L^\infty} + Ce^{s-\tau}s^{3/2} \left\| \frac{q_-(\tau)}{1+|y|^3} \right\|_{L^\infty} + \frac{C}{\sqrt{s}}, \end{array} \tag{34}$$

and

$$\begin{array}{rcl} |\tilde{\alpha}_2(s)| & \leq & \frac{\tau^2}{s^2}|\tilde{q}_2(\tau)| + \frac{CA(s-\tau)}{s^{\alpha+1}}, \\ \left\| \frac{\tilde{\alpha}_-(s)}{1+|y|^3} \right\|_{L^\infty} & \leq & Ce^{-\frac{(s-\tau)}{2}} \left\| \frac{\tilde{q}_-(\tau)}{1+|y|^3} \right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|\tilde{q}_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C}{s^\alpha}, \\ \|\tilde{\alpha}_e(s)\|_{L^\infty} & \leq & Ce^{-\frac{(s-\tau)}{2}} \|\tilde{q}_e(\tau)\|_{L^\infty} + Ce^{s-\tau}s^{3/2} \left\| \frac{\tilde{q}_-(\tau)}{1+|y|^3} \right\|_{L^\infty} + \frac{C}{s^{\alpha-3/2}}. \end{array} \tag{35}$$

(ii) (Nonlinear terms)

$$|\beta_2(s)| \le \frac{(s-\tau)}{s^3},\ |\beta_-(y,s)| \le (s-\tau)(1+|y|^3)s^{-2},\ \|\beta_e(s)\|_{L^\infty} \le (s-\tau)s^{-1/2},$$

$$|\tilde\beta_2(s)| \le (s-\tau)s^{-\alpha-1},\ |\tilde\beta_-(y,s)| \le (s-\tau)(1+|y|^3)s^{-\alpha},\ \|\tilde\beta_e(s)\|_{L^\infty} \le (s-\tau)s^{-\alpha+3/2}.$$

(iii) (Corrective term)

$$|\gamma_2(s)| \le C(s-\tau)s^{-3},\ |\gamma_-(y,s)| \le C(s-\tau)(1+|y|^3)s^{-2},\ \|\gamma_e(s)\|_{L^\infty} \le (s-\tau)s^{-1/2}.$$

(iv) (New term)

$$|\delta_2(s)| \le C(s-\tau)s^{-3},\ |\delta_-(y,s)| \le C(s-\tau)(1+|y|^3)s^{-2}\ ,\|\delta_e(s)\|_{L^\infty} \le C(s-\tau)s^{-1/2}.$$

Proof: We consider, $A \ge 1$, $\tilde A \ge 1$, $\rho > 0$ and $T \le e^{-\rho}$ (so that $s_0 = -\log T \ge \eta$). The terms α, β and γ are already present in the case of the real-valued semilinear heat equation, so we refer to Lemma 3.13 page 167 in [MZ97] for their proof. As for $\tilde\alpha$, since the definition of $\tilde V_{\tilde A}(s)$ is different from the definition of $V_A(s)$, the reader will have absolutely no difficulty to adapt Lemma 3.13 of [MZ97] to the new situation. Thus, we only focus on the new terms $\delta(y,s)$ and $\tilde\beta(y,s)$. Note that since $s_0 \ge \eta$, if we take $\tau \ge s_0$, then $\tau+\eta \le 2\tau$ and if $\tau \le \sigma \le s \le \tau+\eta$, then

$$\frac{1}{2\tau} \le \frac{1}{s} \le \frac{1}{\sigma} \le \frac{1}{\tau}.$$

Let us recall from Bricmont and Kupiainen [BK94] the following estimates on $K(s,\sigma)$, the semigroup generated by $\mathcal{L}+V$:

Lemma 3.8 *(Properties of $K(s,\sigma)$):*
(i) For all $s \ge \sigma > 1$ and y, $x \in \mathbb{R}$

$$|K(s,\sigma,y,x)| \le Ce^{(s-\sigma)\mathcal{L}}(y,x),$$

where $e^{\theta\mathcal{L}}$ is given by

$$e^{\theta\mathcal{L}}(y,x) = \frac{e^\theta}{\sqrt{4\pi(1-e^{-\theta})}}\exp\left[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}\right].$$

(ii) We have for all $s \ge \tau \ge 1$, with $s \le 2\tau$,

$$\left|\int K(s,\tau,y,x)(1+|x|^m)dx\right| \le C\int e^{(s-\tau)\mathcal{L}}(y,x)(1+|x|^m)dx \le e^{s-\tau}(1+|y|^m). \quad (36)$$

Proof:
(i) See page 181 in [MZ97]
(ii) See Corollary 3.14 page 168 in [MZ97]. ■

Estimates of δ defined in (32):
Consider $s \in [\tau,\tau+\eta]$. Since $\tilde q(s) \in \tilde V_A(s)$ by assumption, using (iii) and (iv) of Proposition 3.1, we see that

$$\forall y \in \mathbb{R},\ |\tilde q(y,s)| \le \min\left(\frac{C\tilde A^2}{s^{2-\varepsilon}}(1+|y|^3), \frac{C\tilde A^2}{s^{\alpha-3/2}}\right), \quad (37)$$

hence by definition (19) of N, we obtain

$$\forall y \in \mathbb{R},\ |N(y,s)| \leq C\tilde{A}^4 \min\left(\frac{(1+|y|^3)}{s^{\alpha+\frac{1}{2}-\varepsilon}}, \frac{1}{s^{2\alpha-3}}, \frac{1+|y|^6}{s^{4-2\varepsilon}}\right). \tag{38}$$

Using Lemma 3.8 and the definition (32) of δ, we write

$$\begin{array}{rcl} |\delta(y,s)| & \leq & \displaystyle\int_\tau^s d\sigma \int_{\mathbb{R}} |K(s,\sigma,y,x)N(x,\sigma)|\, dx \\ & \leq & \displaystyle\int_\tau^s d\sigma \int_{\mathbb{R}} e^{(s-\sigma)\mathcal{L}}(y,x)\frac{C\tilde{A}^4(1+|x|^3)}{s^{\alpha+1/2-\varepsilon}}dx \\ & \leq & \displaystyle\frac{C\tilde{A}^4(s-\tau)}{s^{\alpha+1/2-\varepsilon}}e^{s-\tau}(1+|y|^3) \leq \frac{(s-\tau)}{s^2}(1+|y|^3), \end{array} \tag{39}$$

for s_0 large enough, provided that $\varepsilon < 1/2$.
Using the following bounds in (38) and proceeding similarly, we see that

$$\forall y \in \mathbb{R},\ |\delta(y,s)| \leq (s-\tau)\min\left(\frac{1+|y|^3}{s^2}, \frac{1}{\sqrt{s}}, \frac{1+|y|^6}{s^3}\right),$$

provided that $\alpha \geq 2$, $\varepsilon < 1/2$ and s_0 is large enough.
By definition of q_m, q_- and q_e for $m \leq 2$, we write

$$\begin{array}{rcl} |\delta_m(s)| & \leq & \left|\int_{\mathbb{R}} \chi(y,s)\delta(y,s)k_m(y)\rho(y)dy\right| \leq C\int_{\mathbb{R}} |\delta(y,s)|(1+|y|^2)\rho(y)dy \leq \frac{C(s-\tau)}{s^3}. \\ |\delta_-(y,s)| & = & \left|\chi(y,s)\delta(y,s) - \sum_{i=0}^2 \delta_i(s)k_i(y)\right| \leq (s-\tau)(1+|y|^3)\frac{C}{s^2}. \\ \delta_e(y,s)| & = & |(1-\chi(y,s))\delta(y,s)\| \leq (s-\tau)\frac{C}{\sqrt{s}}. \end{array} \tag{40}$$

Estimates of $\tilde{\beta}$ defined in (33):
Consider $s \in [\tau, \tau+\eta]$. Since $q(s) \in V_A(s)$ by assumption, using (i) and (ii) of Proposition 3.1, we see that

$$\forall y \in \mathbb{R},\ |q(y,s)| \leq CA^2 \min\left(\frac{\log s}{s^2}(1+|y|^3), \frac{1}{\sqrt{s}}\right).$$

Using (37) and the definition (19) of $\tilde{B}$, we see that

$$\forall y \in \mathbb{R},\ |\tilde{B}(y,s)| \leq CA^2\tilde{A}^2 \min\left(\frac{\log s}{s^{\alpha+1/2}}(1+|y|^3), \frac{1}{s^{\alpha-1}}, \frac{1+|y|^6}{s^{4-\varepsilon}}\right).$$

Using the definition (33) of $\tilde{\beta}$ and arguing as for estimate (39), we see that

$$\forall y \in \mathbb{R},\ |\tilde{\beta}(y,s)| \leq (s-\tau)\min\left(\frac{1+|y|^3}{s^\alpha}, \frac{1}{s^{\alpha-3/2}}, \frac{1+|y|^6}{s^{\alpha+1}}\right),$$

provided that $\alpha < 3-\varepsilon$ and s_0 is large enough. Arguing as for (40), we get the desired estimates. This concludes the proof of Proposition 3.6 and Proposition 3.3 too. ∎

4 Assymptotic behavior of $u(t)$

We prove Theorem 1 in this section. We will first derive (ii) and (iii) from Section 3, then we will prove (i) and (iv).
Consider $0 < |\tilde{d}_2| \leq 1$. Using Proposition 3.2, Proposition 3.3 and Proposition 3.6, we see that if $A = \tilde{A} = \max(1, A_0, A_4)$, and $T \leq T_6(\tilde{d}_2, A, \tilde{A})$ for some $T_6(\tilde{d}_2, A, \tilde{A}) \leq \min(T_0(A, \tilde{A}), T_1(A, \tilde{A}), T_4(1, A, \tilde{A}))$, then there exists 4 parameters $(d_0, d_1, \tilde{d}_0, \tilde{d}_1)$ such that if $(q(s_0), \tilde{q}(s_0))$ is given by (25), where $s_0 = -\log T$, then

$$\begin{array}{l}\forall s \geq -\log T,\ q(s) \in V_A(s),\ \tilde{q}(s) \in \tilde{V}_{\tilde{A}}(s), \quad \left|\tilde{q}_2'(s) + \frac{2}{s}\tilde{q}_2(s)\right| \leq \frac{\mu_0}{s^{\alpha+1}},\\ \text{with } \mu_0 = \frac{\alpha-2}{4}|\tilde{d}_2|s_0^{\alpha-2},\end{array} \tag{41}$$

and

$$\left|\tilde{q}_2(s_0) - \frac{\tilde{d}_2}{s_0^2}\right| \leq \frac{|\tilde{d}_2|}{4s_0^2}.$$

As announced earlier, we use this property to derive (ii) and (iii) of Theorem 1, then we will prove (i) and (iv).

(ii) This directly follows from (41) with (ii) and (iii) of Proposition 3.1 and selfsimilar transformation (13).
(iii) From (41), we see that

$$\forall s \geq -\log T,\ |\left(s^2\tilde{q}_2\right)'| \leq \frac{\mu_0}{s^{\alpha-1}}, \tag{42}$$

whiich means that $s^2\tilde{q}_2(s)$ has some limits l as $s \to \infty$.
Integrating this inequality between s and $+\infty$, we obtain

$$|s^2\tilde{q}_2(s) - l| \leq \frac{\mu_0}{(2-\alpha)s^{\alpha-2}}. \tag{43}$$

Putting $s = s_0$ in this identity, then using (41), we see that

$$|s_0^2\tilde{q}_2(s_0) - l| \leq \frac{|\tilde{d}_2|}{4} \text{ and } |s_0^2\tilde{q}_2(s_0) - \tilde{d}_2| \leq \frac{\tilde{d}_2}{4},$$

Thus, it follows that

$$|l - \tilde{d}_2| \leq \frac{|\tilde{d}_2|}{2}, \text{ hence } |l| \geq \frac{|\tilde{d}_2|}{2} > 0 \text{ and } l \not\equiv 0.$$

We then write from the decomposition (23) that for all $s \geq -\log T$, $R > 0$ and $|y| \leq R$, $\tilde{q}_e(y, s) = 0$, hence,

$$\tilde{q}(y, s) - \frac{l}{s^2}h_2(y) = \sum_{i=0}^{1}\tilde{q}_i(s)h_i(y) + (\tilde{q}_2(s) - \frac{l}{s^2})h_2(y) + \tilde{q}_-(y, s).$$

Using the fact that for all $s \geq -\log T$, $\tilde{q}(s) \in \tilde{V}_{\tilde{A}}(s)$ (see (41) above), the definition of $\tilde{V}_{\tilde{A}}(s)$ in Proposition 3.1, together with (43), we see that for all $s \geq -\log T,\ R > 0$ and $|y| \leq R$

$$\left|\tilde{q}(y, s) - \frac{l}{s^2}h_2(y)\right|.$$

Using the definition (16) of $\tilde{q}$ and (13) of $\tilde{w}$, we get the desired conclusion.

(i) If $x_0 = 0$, then we see from (9) and (10) that $|v(0,t)| \sim (T-t)^{-1}$ as $t \to T$. Hence u blows up at time T at $x_0 = 0$. It remains to prove that any $a \neq 0$ is not a blow-up point. The following result from Giga and Kohn [GK89] allows us to conclude:

Proposition 4.1 *(Giga and Kohn - No blow-up under some threshold) For all $C_0 > 0$, there is $\eta_0 > 0$ such that if $v(\xi,\tau)$ solves*

$$|v_t - \Delta v| \leq C_0(1 + |v|^p)$$

and satisfies

$$|v(\xi,\tau)| \leq \eta_0 (T-t)^{-1}$$

for all $(\xi,\tau) \in B(a,r) \times [T - r^2, T)$ for some $a \in \mathbb{R}$ and $r > 0$, then v does not blow up at (a,T).

Proof: See Theorem 2.1 page 850 in [GK89]. Note that the proof of Giga and Kohn is valid also when u is complex valued. ■

Indeed, since we see from (9) that

$$\sup_{|x-x_0|\leq |x_0|/2} (T-t)^{-1}|u(x,t)| \leq \left| \varphi\left(\frac{|x_0|/2}{\sqrt{(T-t)|\log(T-t)|}} \right) \right| + \frac{C}{\sqrt{|\log(T-t)|}} \to 0$$

as $t \to T$, x_0 is not a blow-up point of u from Proposition 4.1. This concludes i) of Theorem 1.

(iv)Arguing as Merle did in [Mer92], we derive the existence of a blow-up profile $u^* \in C^2(\mathbb{R}^*)$ such that $u(x,t) \to u^*(x)$ as $t \to T$, uniformly on compact sets of $\mathbb{R}^*$. The profile $u^*(x)$ is not defined at the origin. In the following, we would like to find its equivalent as $x \to 0$ and show that it is in fact singular at the origin. We argue as in Masmoudi and Zaag [MZ08]. Consider $K_0 > 0$ to be fixed large enough later. If $x_0 \neq 0$ is small enough, we introduce for all $(\xi,\tau) \in \mathbb{R} \times [-\frac{t_0(x_0)}{T-t_0(x_0)}, 1)$,

$$V(x_0,\xi,\tau) = (T - t_0(x_0))v(x,t), \tag{44}$$

$$\tilde{V}(x_0,\xi,\tau) = (T - t_0(x_0))\tilde{v}(x,t), \tag{45}$$

$$\text{where, } x = x_0 + \xi\sqrt{T - t_0(x_0)},\ t = t_0(x_0) + \tau(T - t_0(x_0)), \tag{46}$$

and $t_0(x_0)$ is uniquely determined by

$$|x_0| = K_0\sqrt{(T - t_0(x_0))|\log(T - t_0(x_0))|}. \tag{47}$$

From the invariance of problem (2) under dilation, $(V(x_0,\xi,\tau), \tilde{V}(x_0,\xi,\tau))$ is also a solution of (2) on its domain. From (46), (47), (10) and (9), we have

$$\sup_{|\xi|<2|\log(T-t_0(x_0))|^{1/4}} |V(x_0,\xi,0) - f(K_0)| \leq \frac{C}{|\log(T-t_0(x_0))|^{1/4}} \to 0 \text{ as } x_0 \to 0$$

and

$$\sup_{|\xi|<2|\log(T-t_0(x_0))|^{1/4}} \left|\tilde{V}(x_0,\xi,0)\right| \leq \frac{C}{|\log(T-t_0(x_0))|^{1/4}} \to 0 \text{ as } x_0 \to 0.$$

Using the continuity with respect to initial data for problem (2) associated to a space-localization in the ball $B(0, |\xi| < |\log(T - t_0(x_0))|^{1/4})$, we show as in Section 4 of [Zaa98] that

$$\sup_{|\xi|\le|\log(T-t_0(x_0))|^{1/4},\ 0\le\tau<1} |V(x_0,\xi,\tau) - U_{K_0}(\tau)| \le \epsilon(x_0) \text{ as } x_0 \to 0$$
$$\sup_{|\xi|\le|\log(T-t_0(x_0))|^{1/4},\ 0\le\tau<1} |\tilde{V}(x_0,\xi,\tau)| \le \epsilon(x_0) \text{ as } x_0 \to 0,$$

where $U_{K_0}(\tau) = ((1-\tau) + \frac{K_0^2}{8})^{-1}$ is the solution of the PDE (2) with constant initial data $\varphi(K_0)$. Making $\tau \to 1$ and using (46), we see that

$$\begin{array}{rcl} v^*(x_0) = \lim_{t\to T} v(x,t) & = & (T - t_0(x_0))^{-1} \lim_{\tau\to 1} V(x_0, 0, \tau) \\ & \sim & (T - t_0(x_0))^{-1} U_{K_0}(1) \end{array}$$

as $x_0 \to 0$. We note also that

$$|\tilde{v}^*(x_0)| \le \epsilon(x_0)(T - t_0(x_0))^{-1}.$$

Since we have from (47)

$$\log(T - t_0(x_0)) \sim 2\log|x_0| \text{ and } T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0^2|\log|x_0||},$$

as $x_0 \to 0$, this yields (iv) of Theorem 1 and concludes the proof of Theorem 1.

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